The numerator can be evaluated as

$$\int_0^1 Y\ J_o(\beta_n Y)dY = \int_0^{\beta_n} \frac{v}{\beta_n} J_o(v)\frac{dv}{\beta n} = \frac{1}{\beta_n^2}\left[vJ_1\ (v)\right]_o^{\beta n} = \frac{1}{\beta_n^2}[\beta_n\ J_1(\beta_n)]$$

$$\theta_m = 4\sum_{n=1}^{\infty} \frac{e^{-\beta_n^2 X}}{\beta_n^2} \tag{3.60}$$

Again,

$$\frac{d}{dY}\left\{J_0(\beta_n Y)\right\} = -\beta_n J_1(\beta_n Y)$$

$$\left.\frac{d\theta}{dY}\right|_{Y=1} = -2\sum_{n=1}^{\infty} e^{(-\beta_n^2 X)} \tag{3.61}$$

The heat flux from the surface may be calculated as

$$h(T_w - T_m) = k\frac{\partial T}{\partial r}\ \Bigg|_{r=r_0} = \frac{k(T_\infty - T_w)}{r_o}\left.\frac{d\theta}{d\left(\frac{r}{r_o}\right)}\right|_{Y=1} \tag{3.62}$$

$$\frac{hD}{k} = \frac{2\times 2\sum_{n=1}^{\infty} e^{-\beta_n^2 X}}{\theta_m} = \frac{\sum_{n=1}^{\infty} e^{-\beta_n^2 X}}{\sum_{n=1}^{\infty}(e^{-\beta_n^2 X})/\beta_n^2} \tag{3.63}$$

For β_1, $\frac{hD}{k} = \frac{e^{-\beta_1^2 X}\ \beta_1^2}{e^{-\beta_1^2 X}} = \beta_1^2$ or

$$Nu_D = 5.7831 \tag{3.64}$$

3.4.3 Heat Transfer in Circular Tube for Hydrodynamically and Thermally Developed Flow with Uniform Wall Temperature (UWT) Condition

The governing equation is

$$\rho c_p u\frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3.65}$$

$\theta(r) = \frac{T-T_w}{T_c-T_w}$, where T_c is the centerline temperature and T_w is the wall temperature; introducing $\eta = \frac{r}{r_o}, z = \frac{x}{r_0}, U(\eta) = \frac{u}{u_m}$.

We get

$$\frac{\rho c_p u_m U(\eta)}{r_0}\frac{\partial T}{\partial z} = \frac{k}{r_0^2}\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial T}{\partial \eta}\right)$$

$$\left(\frac{\rho\nu c_p}{k}\right)\frac{u_m r_0}{\nu}U\frac{\partial T}{\partial z} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial T}{\partial \eta}\right)$$

$$\frac{PrRe_D}{2}U\frac{\partial T}{\partial z} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial T}{\partial \eta}\right)$$

Now, $T = T_w + (T_c - T_w)\theta(r)$; $T_c \equiv f_n(z)$

$$\frac{\partial T}{\partial z} = \theta(r)\frac{\partial T_c}{\partial z}$$

$$\frac{\partial T}{\partial r} = (T_c - T_w)\frac{\partial \theta}{\partial r}$$

$$\frac{\partial T}{\partial \eta} = (T_c - T_w)\frac{\partial \theta}{\partial \eta}$$

On substitution,

$$Pe\ \frac{U\theta}{2}\ \frac{\partial T_c}{\partial z} = \frac{1}{\eta}\ \frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right)(T_c - T_w)$$

or

$$\left[\frac{1}{T_c(z) - T_w}\right] Pe\ \frac{dT_c}{dz} = \frac{\frac{1}{\eta}\ \frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right)}{\frac{U(\eta)}{2}\theta(\eta)} = -\lambda^2 \quad \text{(constant)} \tag{3.66}$$

$$\frac{Pe\ dT_c/dz}{T_c - T_w} = -\lambda^2$$

$$T_c(z) = T_w + Ce^{-\lambda^2 z/Pe}$$

The second equation gives

$$\frac{1}{\eta}\ \frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right) + \frac{\lambda^2}{2}U\theta = 0$$

where, $\frac{u}{u_m} = 2\left[1 - \left(\frac{r}{r_0}\right)^2\right] = 2[1 - \eta^2]$ or

$$\theta'' + \frac{1}{\eta}\theta' + \lambda^2(1 - \eta^2)\theta = 0 \tag{3.67}$$

The equation is to be solved using the following boundary conditions, *at* $r = 0$, $T = T_c$ which means at $\eta = 0$, $\theta = 1$ at $r = r_0$, $T = T_w$ which means at $\eta = 1$, $\theta = 0$. The solution of the above equation is an infinite series.

Let the general solution be

$$\theta = \sum_{n=0}^{\infty} C_n \eta^n$$

Now,

$$\begin{aligned}\theta &= C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 + \cdots + C_n\eta^n \\ \text{then } \theta' &= C_1 + 2\ C_2\eta + 3\ C_3\eta^2 + \cdots + (n+1)\ C_{n+1}\eta^n \\ \text{and } \theta'' &= 2\ C_2 + 6C_3\eta + \cdots + (n+1)n\ C_{n+1}\ \eta^{n-1} + (n+2)(n+1)C_{n+2}\ \eta^n.\end{aligned}$$

Plugging in to the original equation, one gets

$$\begin{aligned}& 2\ C_2 + 3\cdot 2\ C_3\eta + \cdots + (n+2)(n+1)\ C_{n+2}\eta^n \\ + \ & \frac{C_1}{\eta} + 2\ C_2 + 3\ C_3\eta + \cdots + (n+1)\ C_{n+1}\eta^{n-1} + (n+2)\ C_{n+2}\eta^n \\ + \ & \lambda^2 C_0 + \lambda^2 C_1\eta + \lambda^2 C_2\eta^2 + \cdots + \lambda^2 C_n\eta^n \\ = \ & \lambda^2 C_0\eta^2 + \lambda^2 C_1\eta^3 + \lambda^2 C_2\eta^4 + \cdots + \lambda^2 C_n\eta^{n+2}\end{aligned}$$

Equating the equal powers of η

$$\begin{aligned}& C_1 = 0 \\ & 2.1C_2 + 2C_2 + \lambda^2 C_0 = 0 \\ & 3.2C_3 + 3C_3 + \lambda^2 C_1 = 0 \\ & 4.3C_4 + 4C_4 + \lambda^2 C_2 = \lambda^2 C_0 \\ & 5.4C_5 + 5C_5 + \lambda^2 C_3 = \lambda^2 C_1 \\ & \vdots \\ & (n+2)(n+1)C_{n+2} + (n+2)C_{n+2} + \lambda^2 C_n = \lambda^2 C_{n-2}\end{aligned}$$

As $C_1 = 0, C_3, C_5 \ldots$ all odd coefficients are zero.
Using $n = 2m$,

$$\begin{aligned}& (2m+2)(2m+1)C_{2m+2} + (2m+2)C_{2m+2} + \lambda^2 C_{2m} = \lambda^2 C_{2m-2} \\ & (2m+2)^2 C_{2m+2} + \lambda^2 C_{2m} = \lambda^2 C_{2m-2} \\ & C_{2m+2} = \frac{\lambda^2}{(2m+2)^2}(C_{2m-2} - C_{2m}) \\ & C_{2m} = \frac{\lambda^2}{(2m)^2}(C_{2m-4} - C_{2m-2})\end{aligned}$$

Therefore, we can write

$$\theta = \sum_{n=0}^{\infty} C_{2n}\eta^{2n} \tag{3.68}$$

where $\quad C_{2n} = \dfrac{\lambda^2}{(2n)^2}[C_{2n-4} - C_{2n-2}] \quad$ for $\quad n \geq 2$

We can also write

$$\theta = \sum_{n=0}^{\infty} C_{2n}\eta^{2n} = C_0 + C_2\eta^2 + C_4\eta^4 + C_6\eta^6 + \cdots$$

Invoking the boundary condition, at $\eta = 0, \theta = 1$ we get

$$C_0 = \theta_{\text{at } \eta=0} = 1$$

We also know at $r = r_0$, i.e., at $\eta = 1, \theta = 0$

$$C_0 + C_2 + C_4 + C_6 + \ldots = 0$$

$$C_0 - \frac{\lambda^2}{4}C_0 + \frac{\lambda^2}{16}\left(1 + \frac{\lambda^2}{4}\right)C_0 - \frac{\lambda^2}{36}\left(\frac{5\lambda^2}{16} + \frac{\lambda^4}{64}\right)C_0 + \cdots = 0$$

$$C_0\left[1 - \frac{\lambda^2}{4} + \frac{\lambda^2}{16} + \frac{\lambda^4}{64} - \frac{5\lambda^4}{36 \times 16} - \frac{\lambda^6}{36 \times 64}\right] + \cdots = 0$$

Utilizing the fourth power of the series,

$$C_0\left[1 - \frac{\lambda^2}{4} + \frac{\lambda^2}{16} + \frac{\lambda^4}{64} - \frac{5\lambda^4}{576}\right] = 0$$

Since $C_0 \neq 0$,

$$1 - \frac{\lambda^2}{4} + \frac{\lambda^2}{16} + \frac{\lambda^4}{144} = 0$$

$$\lambda^4 - 27\lambda^2 + 144 = 0$$

Solving numerically by Newton Rapshon method,

$$\lambda = 2.704364 \tag{3.69}$$

3.4.4 Calculation of Nusselt Number for UWT

From the governing equation we get

$$\int_0^{r_0} \rho c_p u \frac{\theta}{r_0}\frac{dT_c}{dz} r dr = k r_0 \frac{\partial T}{\partial r}\,|_{r=r_0}$$

or

$$\int_0^{r_0} k\frac{\nu}{\alpha}\,\frac{u_m 2r_0}{\nu}\,\frac{r}{2r_0^2}\,\frac{u}{u_m}\,(T - T_w)\,\frac{dT_c/dz}{(T_c - T_w)}\,dr = r_0 q_w''$$

$$\int_0^{r_0} \frac{PrRe_D\frac{dT_c}{dz}}{(T_c - T_w)}\,k\,\frac{r}{2r_0^2}\,\frac{u}{u_m}\,(T - T_w)\frac{2\pi}{2\pi}dr = r_0 q_w''$$

or

$$\int_0^{r_0} -\frac{\lambda^2 k}{2}\,\frac{2\pi u r (T - T_w) dr}{2\pi u_m r_0^2} = r_0 q_w''$$

or

$$-\frac{\lambda^2 k}{4}\left\{\frac{\int_0^{r_0} uT2\pi r dr}{u_m \pi r_0^2} - \frac{T_w \int_0^{r_0} u2\pi r dr}{u_m \pi r_0^2}\right\} = r_0 q_w''$$

or

$$-\frac{\lambda^2 k}{4}(T_m - T_w) = r_0 q_w''$$

or

$$\frac{q_w''}{(T_w - T_m)}\frac{2r_0}{k} = \frac{hD}{k} = \frac{\lambda^2}{2}$$

The Nusselt number for hydrodynamically and thermally fully developed flow (subject to uniform wall temperature) is:

$$Nu_D = \frac{\lambda^2}{2} = 3.656 \tag{3.70}$$

For parallel plate channels, Nusselt number for hydrodynamically and thermally fully developed flow is

$$Nu = 7.5407 \text{ (uniform wall temperature)}$$
$$Nu = 8.2352 \text{ (uniform wall heat flux)}$$

In the above two expressions, the Nusselt number has been calculated based on the characteristic length $2b$, where b is the channel height.

3.5 GRAETZ PROBLEM

The problem addresses hydrodynamically developed, thermally developing flow subject to uniform wall heat flux as shown in Fig. 3.11 and is known as the Graetz problem [2]. The problem was further extended by Sellars et al. [3].

For $x < 0$: Both the fluid and the wall have uniform temperature, say T_0.
For $x > 0$: Uniform wall heat flux is applied.

We start with a situation which prevails at a very large x. We refer to hydrodynamically and thermally fully developed flow with uniform wall heat flux.
We have already found the solution as

$$T(x,r) = T_w - \frac{2u_m r_0^2}{\alpha}\left(\frac{dT_m(x)}{dx}\right)\left[\frac{3}{16} + \frac{1}{16}\left(\frac{r}{r_0}\right)^4 - \frac{1}{4}\left(\frac{r}{r_0}\right)^2\right]$$

After substitution of $\left(\frac{dT_m}{dx}\right)$

$$T(x,r) = T_w + \frac{q_w'' r_0}{k}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{3}{4}\right] \tag{3.71}$$

3.6 SUMMARY OF SOLUTIONS FOR VARIOUS FLOW AND HEAT TRANSFER SITUATIONS IN PIPE FLOWS

A marvelous compilation of different results due to various investigations are available in Shah [4] and Shah and London [7]. Fig. 3.12 provides the results in a comprehensive manner. Nu_D is plotted against the dimensionless parameter $\frac{x/D}{Re\ Pr}$ or

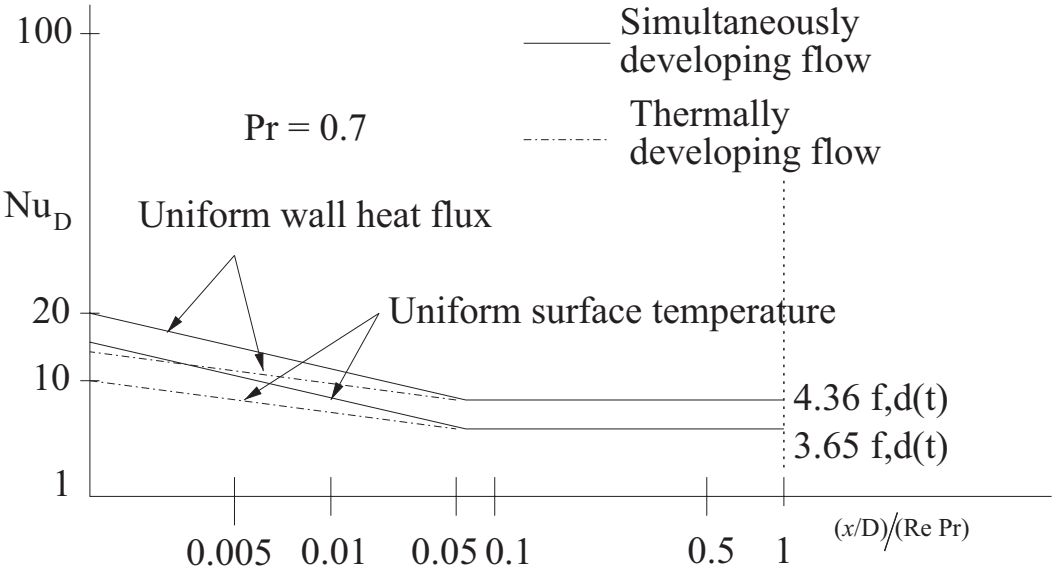


Figure 3.12 Variation of Nusselt number in a circular duct

reciprocal of Graetz number ($G \equiv \frac{D\ Re\ Pr}{x}$). Fully developed values are independent of Prandtl number. Fully developed conditions are reached for $\frac{x/D}{Re\ Pr} \approx 0.05$.

For the constant surface temperature condition, it is desirable to know the average convection coefficient (in the entry length) as $q_{conv} = \overline{h}\ A_w\ \Delta T_{l,m}$. Hausen presents the following correlation for hydrodynamically fully developed flow (laminar) in tubes at constant wall temperature

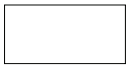
$$\overline{Nu}_D = 3.66 + \frac{0.0668(D/L)Re_D\ Pr}{1 + 0.04[(D/L)Re_D\ Pr]^{2/3}}$$

Because the above result is for thermally developing flow, it is not generally applicable. For simultaneously developing flow, a suitable correlation due to Sieder and Tate [8] is of the form:

$$\overline{Nu}_D = 1.86\left[\frac{Re_D\ Pr}{L/D}\right]^{1/3}\ \left(\frac{\mu}{\mu_w}\right)^{0.14}$$

All properties are evaluated at $T_m = (T_{m,0} + T_{m,i})/2$; μ_w is the viscosity of the fluid at wall temperature.

Table 3.2
Nusselt numbers for fully developed laminar flow in tubes of differing cross sections

Cross section	$\frac{b}{a}$	$Nu_D = (hD_h)/k$	
		uniform q''_w	uniform T_w
	–	4.36	3.66
b a	1.0	3.61	2.98
b a	1.43	3.73	3.08
b a 	2.0	4.12	3.39
b a	3.0	4.79	3.96
"	4.0	5.33	4.44
"	8.0	6.49	5.60
(parallel plate)	∞	8.23	7.54
		3.00	2.35

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